An invariant variational principle for Hamiltonian mechanics

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Abstract

It is shown that the action for Hamiltonian equations of motion can be brought into invariant symplectic form. In other words, it can be formulated directly in terms of the symplectic structure ω without any need to choose some 1-form γ, such that $\omega = \mathrm{d}\gamma$, which is not unique and does not even generally exist in a global sense.

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1. Introduction

Hamiltonian mechanics is defined by a smooth function (Hamiltonian) on an even-dimensional manifold M^{2n} equipped with a symplectic structure, i.e. a closed non-degenerate differential form,

$$\omega = \sum_{\mu > \nu} \omega_{\mu\nu}(x)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2} \sum_{\mu,\nu} \omega_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \qquad x \in M^{2n}.$$

Using Darboux transformation these coordinates may be divided into two groups, $x = (q^1, \ldots, q^n, p_1, \ldots, p_n)$ with q^i and p_i being generalized coordinates and momenta respectively, but this transformation is not unique and in general can be performed only locally.

Equations of motion are given by Poisson brackets

$$\dot{x}^\mu = \{x, H\} = \sum_{\nu=1}^{2n} \omega^{\mu\nu}(x)\partial_\nu H, \tag{1}$$

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where the right-hand side can be regarded as a vector obtained by contraction of Poisson bivector (dual to ω) with the differential form dH; in local coordinates the bivector is represented by $\omega^{\mu\nu}$, the matrix inverse to $\omega_{\mu\nu}$. In the case of the standard symplectic structure, i.e.

$$\omega^{\mu\nu} = \begin{pmatrix} O_n & E_n \\ -E_n & O_n \end{pmatrix},$$

where E_n and O_n are the unit and the zero matrices respectively, a simple action exists for these equations of motion [1]:

$$S = \int_{t_1}^{t_2} \left(\sum_{\alpha=1}^{n} p_\alpha \dot{q}^\alpha - H \right) dt, \tag{2}$$

with the boundary conditions $q(t_1) = q_1, q(t_2) = q_2$.

But expression (2) does not seem satisfactory from a geometric point of view because it is suitable only in Darboux coordinates and does not use the symplectic form explicitly. So, our goal would be to find an invariant geometric form of the action. In sections 2 and 3 we discuss dynamical systems with exact symplectic forms and, first, the very important case of 'classical' Hamiltonian systems (i.e. those in $\mathbb{R}^{2n}$ with the standard symplectic structure). The special emphasis is placed on the problem of boundary conditions [2]. After that we explore non-exact symplectic forms (see also [3] for Kähler manifolds as phase spaces, which are unitary phase spaces in the terminology of [3]). To the best of our knowledge there is no invariant variational principle in the literature for the general case. However, such systems may be of interest for different reasons. For example, systems with gyroscopic forces cannot be presented in a straight-forward way as Hamiltonian systems with one-valued Hamiltonians [4]. The problem can be solved [4, 5] by some non-exact symplectic structure. In particular, the so-called Kirchhoff-type systems [5] can be brought into Hamiltonian form on a cotangent bundle over 2-sphere with a volume form of the configuration space $\mathcal{S}^2$ being added to the standard symplectic structure. It is enough to mention that systems of this kind include [5] a rigid body in an ideal incompressible fluid at rest at infinity, a rigid body with a fixed point in axially symmetric potential field, spin dynamics in the A-phase of the superfluid ^{3}He (Leggett equations), etc, and at the quantum level we would have a sphere with non-commutative coordinates. Motivated by these reasons, in sections 4 and 5 we establish an invariant variational principle for arbitrary Hamiltonian systems. Section 6 is devoted to some examples and discussions.

2. Brief review of known results

As it was mentioned above, for classical Hamiltonian systems one can use the non-invariant variational principle with action (2) to obtain the equations of motion (1). This action can be brought into invariant form as follows [6, 7]:

$$S = \int_{t_1}^{t_2} \left(\frac{1}{2} \omega_{\mu\nu} \dot{x}^\nu x^\mu - H \right) dt = \int_{t_1}^{t_2} \left(\frac{1}{2} (p_i \dot{q}^i - \dot{p}_i q^i) - H \right) dt, \tag{3}$$

$\mu, \nu = 1, \ldots, 2n, i = 1, \ldots, n$, where the summation over repeated indices is assumed and Dirichlet boundary conditions ($\delta q(t_1) = \delta q(t_2) = \delta p(t_1) = \delta p(t_2) = 0$) on both coordinates and momenta variations are imposed, which is two times as much as the number of possible independent boundary conditions on the values of coordinates and momenta themselves in Cauchy or boundary problem for the equations of motion. We have a mismatch between the number of required variational boundary conditions and the number of independent solutions of equations of motion.

So, the Hamiltonian variational principle differs in this sense seriously from the Lagrangian one. Actually, one may suspect that it can be formulated in a more usual way by fixing the values of the functions $\phi_i = \arctan \frac{q_i}{p_i}$ at the boundary points. Indeed, variations of ϕs equal to

$$\delta\phi_i = \frac{p_i \delta q_i - q_i \delta p_i}{q_i^2 + p_i^2};$$

and $\delta\phi_i = 0$ at $t = t_0$ and $t = t_1$ is exactly what we need to make the boundary terms in the first variation of (3) equal to zero. Equations $\phi_i = $ const define some n-dimensional submanifolds in $2n$-dimensional phase space. And it would be curious to introduce new pairs of coordinates in the following way:

$$q_i = r_i \sin \phi_i, \qquad p_i = r_i \cos \phi_i.$$

With the definition $P_i = r_i^2/2$ it is easy to get $\frac{1}{2}(p_i \dot{q}_i - q_i \dot{p}_i) = P_i \dot{\phi}_i$ and

$$S = \int_{t_1}^{t_2} \left(\sum_i P_i \dot{\phi}_i - H(\phi, P) \right) dt.$$

It looks quite good. But the point is that these variables are nothing more than a particular choice of Darboux coordinates. We could well have started with action (3) for these canonical variables and considered other preferred functions of the form $\tilde{\phi}_i = \arctan \frac{\phi_i}{P_i}$. Hence we should stress once more that there is *no invariant way* to fix only a half of boundary conditions without sticking to some coordinate choice, because what we have done here is just a canonical transformation to new variables, P_i and ϕ_i.

It is of crucial importance for the geometric meaning of action (3) that the phase space is $\mathbb{R}^{2n}$ because the radius vectors are used. (But the symplectic form could be arbitrary.) However, there is a natural analog of action (2) which is valid for any exact symplectic manifold. (Note still that for any compact symplectic manifold (without boundary) ω is always *not* exact [8].)

Suppose that the symplectic form is exact: $\omega = d\gamma$, where γ is some 1-form (for classical systems $\gamma = p_i \, dq^i$). (Even if the symplectic form is not globally exact, such 1-forms always exist locally.) Then the matrix $\omega_{\mu\nu}$ can be expressed in terms of the components of $\gamma = A_\mu \, dx^\mu$: $\omega_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. And the action can be written as follows [2]:

$$S = \int_{x_1}^{x_2} \gamma - \int_{t_1}^{t_2} H \, dt = \int_{t_1}^{t_2} (A_\mu \dot{x}^\mu - H) \, dt. \qquad (4)$$

Actually, action (4) is just a special case of Birkhoff theory (Pfaff–Birkhoff variational principle) [3, 9, 10]

$$S = \int_{t_1}^{t_2} (A_\mu(x, t)\dot{x}^\mu + B(x, t)) \, dt$$

with B staying for Hamiltonian. The equations of motion should be derived from the variational principle with Dirichlet boundary conditions. This formalism is valid also for non-conservative systems with a time-dependent symplectic structure.

The variation of action (4) yields Hamiltonian equations of motion if one imposes Dirichlet boundary conditions as it was done above in $\mathbb{R}^{2n}$ with the same problems persisting. In principle, provided that we know an explicit form of A_μ we can introduce an analog of the functions ϕ. But if we want to allow for different possible choices of γ then we are forced to admit the whole set of boundary conditions, $\delta q(t_1) = \delta q(t_2) = \delta p(t_1) = \delta p(t_2) = 0$.

This mismatch between the number of boundary conditions for the variational problem and the number of boundary conditions for the equations of motion often makes physicists to worry

about more that it really deserves. The habit to identify one kind of boundary conditions with another is so strong that many distinguished authors [11–13] do not fix initial and final points for action (2) completely but impose only one half of the conditions ($\delta q(t_1) = \delta q(t_2) = 0$), even if they are going to discuss canonical transformations in the same text.

Actually, the story of this approach goes back to the early days of Hamiltonian mechanics. But already in the book by Poincaré [14] a discussion of canonical transformations appeared with the conclusion that such principle has different forms in different canonical coordinates (in the sense of different boundary conditions). For this reason it is commonly accepted in the mathematical literature to fix the boundary points completely by $4n$ conditions [15], thus providing the variational principle with a kind of invariance. And it is pleasant to note that the same is done in the classical textbook for physicists [16] (and in some other physical books [17]) where it is also stressed that only the whole set of boundary conditions allows one to add a total time derivative term $\frac{\mathrm{d}}{\mathrm{d}t} f(p, q)$ to the integrand in (2).

3. The case of 'wrong' boundary conditions

In this section we would like to add several comments and to give a variational characteristic of correctly chosen boundary points, which is in a sense obvious and presumably not easy to use but, may be, worth mentioning still. Some other problems would be discussed in a forthcoming article by L V Prokhorov and A S Ushakov.

In any case the correct statement is that the physical trajectory brings the action to the stationary value in the class of trajectories with fixed boundary points in the phase space. But an important fact about the variational principles is that one would probably wish to use the relevant principle in order to get the equations of motion or to apply it instead to the equations if it can make the mathematics easier. At this point we encounter with a kind of problem: the number of boundary conditions required is greater than the number of Hamiltonian equations of motion. And for every initial point at t_1 there is only one final point at t_2 for which the extremizing trajectory exists. To get the physical trajectory one needs to guess the unique final point from the continuum of all the phase space points. But this problem cannot lead to a false trajectory because if the final point is chosen wrong the equations of motion cannot be satisfied and the action has no critical point in our class of variations. One could even say that it is not a problem at all because one can choose arbitrary initial point and fix the final point formally without any idea of where it is fixed [2]; it will allow him to get the equations of motion and find the location of the final point after that.

The well-known and very important property of action (2) is that it is bounded neither from below nor from above. As a consequence, the physical trajectory is not a kind of extremum but rather is a saddle point. In particular, it means that the direct methods of calculus of variations cannot be directly applied. And it was not until about 30 years ago that certain progress has been made in this direction [6, 18–20] due to the technique of approximation by some finite-dimensional critical point problem [18]. It means also that if one shifts a final point a little bit from its correct position then the range of the action does not change radically, just its saddle point is gone.

We can regard the first variation of action (2) as a linear operator acting from L^2 space of functions $\delta x(t) : [t_1, t_2] \to \mathbb{R}^{2n}$ to $\mathbb{R}$, which depends on a chosen path from $x(t_1)$ to $x(t_2)$. Its operator norm equals

$$\|\hat{\delta}_S\| = \sqrt{\int_{t_1}^{t_2} \mathrm{d}t \sum_i \left(\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right)^2 + \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right)^2 \right)}.$$

And only for the physical trajectory we have $\|\hat{\delta}_S\| = 0$, otherwise $\|\hat{\delta}_S\| > 0$. If we have a 'wrong' final point then for all possible paths $\|\hat{\delta}_S\| > 0$. Moreover, the minimal value of $\|\hat{\delta}_S\|$ normally exists, so that it is separated from 0.

The minimal path for the functional $\|\hat{\delta}_S\|$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) = -\frac{\partial^2 H}{\partial q_i\,\partial p_k}\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right) + \frac{\partial^2 H}{\partial q_i\,\partial q_k}\left(\dot{p}_k + \frac{\partial H}{\partial q_k}\right),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right) = \frac{\partial^2 H}{\partial q_i\,\partial p_k}\left(\dot{p}_k + \frac{\partial H}{\partial q_k}\right) - \frac{\partial^2 H}{\partial p_i\,\partial p_k}\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)$$

which allows exactly $4n$ independent boundary conditions. If these conditions are 'right' we also have $\dot{q}_i - \frac{\partial H}{\partial p_i} = 0$, $\dot{p}_i + \frac{\partial H}{\partial q_i} = 0$ and $\|\hat{\delta}_S\| = 0$. In other cases $\|\hat{\delta}_S\| > 0$. Let us consider a very simple example with $n = 1$ and $H = \frac{p^2}{2}$. It implies $\|\hat{\delta}_S\|^2 = \int_{t_1}^{t_2} \mathrm{d}t((\dot{q} - p)^2 + \dot{p}^2)$. The minimum is given by $q(t) = \frac{C_1}{6}t^3 + \frac{C_2}{2}t^2 + (C_3 - C_1)t + C_4$ and $p(t) = \frac{C_1}{6}t^2 + C_2 t + C_3$. For the initial point $(q_0, p_0 \neq 0)$ we have to choose $(q_0 + p_0(t_2 - t_1), p_0)$ as the final point. But if we take $(q_0 + \alpha p_0(t_2 - t_1), p_0 + \beta)$, the constants C_i would be $C_4 = q_0$, $C_3 = p_0$, $C_1 = \frac{6\beta + 12(\alpha - 1)p_0}{(t_2 - t_1)^2 + 6(t_2 - t_1)}$, $C_2 = \frac{\beta}{t_2 - t_1} - \frac{C_1(t_2 - t_1)}{2}$ and finally $\|\hat{\delta}_S\|^2 = C_1^2\left(\frac{7}{12}(t_2 - t_1)^3 + (t_2 - t_1)\right) + \frac{\beta^2}{t_2 - t_1}$. Only for $\alpha - 1 = \beta = 0$ do we have $\|\hat{\delta}_S\| = 0$.

Of course, we used here even more complicated equations of motion than the original ones. But the variational problem is quite different from that of (2) because $\|\hat{\delta}_S\|$ has a unique global minimum, and its value in principle can be found by direct methods of variational calculus. After that, these minimal values can be considered as a function of the final point (the initial point is fixed) which, in turn, also has a unique global minimum at the 'right' final point. The minimal value is zero.

Note that if the symplectic manifold is different from $\mathbb{R}^{2n}$ we may consider the variations as elements of the tangent space. We do not want to go into any details here but for any symplectic manifold there exists (not unique) an almost complex structure and a Riemannian metric which are in a sense compatible with ω (see, for example, [20], p 14), and it makes a room for the constructions explained above to be used in the general setting of action (4).

4. Invariant variational principle

In the general case one should use the symplectic form ω explicitly. And all the vectors we have are in the tangent space. It means that the surface of integration should be two dimensional. Looking at formulae (3) and (4) we expect that the action should have the following form:

$$S = \int_\sigma (\omega - \mathrm{d}H \wedge \mathrm{d}t) = \int_\sigma \left(\frac{1}{2}\omega_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu - \mathrm{d}H \wedge \mathrm{d}t\right). \qquad (5)$$

More precisely, we consider a trivial bundle over M^{2n}: $F^{2n+1} = M^{2n} \times \mathbb{R}$ (extended phase space) and endow it with a new coordinate t (time) so that the basis of 1-forms gains one more, $(2n + 1)$th, element, $\mathrm{d}t$, with an axiom $\mathrm{d}t \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}t$. The differential forms ω and $\mathrm{d}H$ are defined in F simply by their coordinate expressions in M which may be invariantly interpreted as a pullback of these forms generated by the natural projection of F to M. Any one-parameter family of initial points $x(\varepsilon, t_1)$, $\varepsilon \in [0, E]$ defines a two-dimensional surface σ of trajectories in F with the curve of the final points at $t = t_2$. It is important that the initial and final curves (γ_1 and γ_2) should be transversal to the physical trajectories for the surface σ to be well defined. One possible way to ensure it is to choose the curves transversal to the hypersurfaces of constant Hamiltonian. In this case the parameter ϵ also
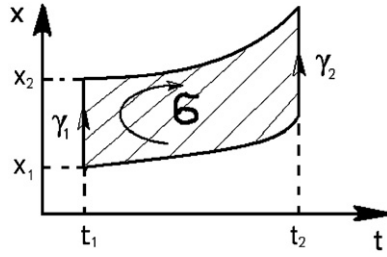
Figure 1. Surface of integration.

gains a possible interpretation as a value of the Hamiltonian function, so that one considers a family of trajectories with different amounts of energy. In the remainder of the section we prove under these assumptions the following.

Theorem:

(1) The surface σ is a stationary one for action (5) in the class of smooth surfaces $x = x(\varepsilon, t), \varepsilon \in [0, E], t \in [t_1, t_2]$ with fixed ends $\delta x(\varepsilon, t_1) = \delta x(\varepsilon, t_2) = 0, \forall \epsilon$ (figure 1).

(2) An arbitrary surface $\tilde{\sigma}$ from this class is stationary if and only if its boundary trajectories $x(0, t)$ and $x(E, t)$ are physical, i.e. they satisfy the equations of motion.

Let us start with a simple case when the surface σ and small variations of it belong altogether to one coordinate chart of the phase space manifold. Strictly speaking, it means that we use only a contractible domain in M^{2n} and there does exist some suitable 1-form γ. From its definition $\omega = d\gamma$ it follows that action (5) equals action (4) on $\partial\sigma$ and the statement is trivial. Nevertheless we want to proceed with explicit calculations in order to make an illustration of how it works if one does not know an appropriate 1-form γ. We perform the variations here in a somewhat formal way. We vary coordinates even under the differential symbols in differential forms as if they were just ordinary functions. It can be justified if we take into account that for the small variations of the surface there is a natural one-to-one correspondence between points of the initial and the final surfaces. It allows us to vary the integrand instead of the domain of integration. It is also important to mention that we consider only continuous variations of the surface σ so that all surfaces are homotopic to each other and Stokes' theorem is applicable.

The variation of action (5) is equal to

$$\delta S = \int_{\sigma} \left(\frac{1}{2}\partial_\alpha \omega_{\mu\nu}\delta x^\alpha \, dx^\mu \wedge dx^\nu + \frac{1}{2}\omega_{\mu\nu}\partial_\alpha(\delta x^\mu) \, dx^\alpha \wedge dx^\nu \right.$$
$$\left. + \frac{1}{2}\omega_{\mu\nu}\partial_\alpha(\delta x^\nu) \, dx^\mu \wedge dx^\alpha - \partial_\beta(\partial_\alpha H \delta x^\alpha) \, dx^\beta \wedge dt \right).$$

We note that the second and third terms under the integral are equal, and integration by parts gives

$$\delta S = \int_{\sigma} \left(\frac{1}{2}\partial_\alpha \omega_{\mu\nu}\delta x^\alpha \, dx^\mu \wedge dx^\nu + (\partial_\alpha(\omega_{\mu\nu}\delta x^\mu) - \partial_\alpha \omega_{\mu\nu}\delta x^\mu) \, dx^\alpha \wedge dx^\nu \right.$$
$$\left. - \partial_\beta(\partial_\alpha H \delta x^\alpha) \, dx^\beta \wedge dt \right).$$

Using the Jacobi identify

$$\partial_\alpha \omega_{\mu\nu} = -\partial_\nu \omega_{\alpha\mu} - \partial_\mu \omega_{\nu\alpha} = -\partial_\nu \omega_{\alpha\mu} + \partial_\mu \omega_{\alpha\nu}$$

we get

$$\delta S = \int_\sigma \left(\frac{1}{2} \partial_\mu \omega_{\alpha\nu} \delta x^\alpha \, dx^\mu \wedge dx^\nu - \frac{1}{2} \partial_\nu \omega_{\alpha\mu} \delta x^\alpha \, dx^\mu \wedge dx^\nu \right.$$

$$\left. + (\partial_\alpha (\omega_{\mu\nu} \delta x^\mu) - \partial_\alpha \omega_{\mu\nu} \delta x^\mu) \, dx^\alpha \wedge dx^\nu - \partial_\beta (\partial_\alpha H \delta x^\alpha) \, dx^\beta \wedge dt \right)$$

$$= \int_\sigma \left(-\frac{1}{2} \partial_\mu \omega_{\alpha\nu} \delta x^\alpha \, dx^\mu \wedge dx^\nu + \frac{1}{2} \partial_\nu \omega_{\alpha\mu} \delta x^\alpha \, dx^\nu \wedge dx^\mu \right.$$

$$\left. + \partial_\alpha (\omega_{\mu\nu} \delta x^\mu) \, dx^\alpha \wedge dx^\nu - \partial_\beta (\partial_\alpha H \delta x^\alpha) \, dx^\beta \wedge dt \right)$$

$$= \int_\sigma \left(\frac{1}{2} \partial_\alpha (\omega_{\mu\nu} \delta x^\mu) \, dx^\alpha \wedge dx^\nu - \partial_\alpha (\partial_\beta H \delta x^\beta) \, dx^\alpha \wedge dt \right)$$

$$= \int_{\partial\sigma} (\omega_{\mu\nu} \delta x^\mu \, dx^\nu - \partial_\beta H \delta x^\beta \, dt)$$

$$= \int_{\partial([0,E] \times [t_1, t_2])} (\omega_{\mu\nu} \delta x^\mu \, dx^\nu - \partial_\mu H \delta x^\mu \, dt)$$

$$= \int_{t_1}^{t_2} (\omega_{\mu\nu} \dot{x}^\nu (E, t) - \partial_\mu H(x(E, t))) \delta x^\mu \, dt$$

$$- \int_{t_1}^{t_2} (\omega_{\mu\nu} \dot{x}^\nu (0, t) - \partial_\mu H(x(0, t))) \delta x^\mu \, dt.$$

The variations δx^μ are arbitrary and can be performed independently at $\epsilon = 0$ and $\epsilon = E$. Hence, $\delta S = 0$ is equivalent to Hamiltonian equations of motion $\omega_{\mu\nu} \dot{x}^\nu = \partial_\mu H$ for $x(0, t)$ and $x(E, t)$. It means that in the space of all possible one-parameter families of curves we have an 'infinite-dimensional stationary manifold' for action (5). The intermediate trajectories do not influence the stationarity condition because the differential form under the integral in (5) is closed and any variation within one homotopy class with the whole boundary $\partial\sigma$ fixed is subject to Stokes' theorem.

It is worth mentioning that before the last step of the calculation all x and δx could be regarded as functions of point in F^{2n+1}. Only at the last step we restrict ourselves to the boundary lines and consider these functions as well-defined functions of time. It means that we could vary the time in (5) independently. It is easy to see that the consequence of such variation is that the Hamiltonian does not change with time (along the physical trajectory). It follows also from the equations of motion. It is not surprising that we have got nothing new because the variation of time is equivalent to some variation of dynamical functions $x(t)$.

Suppose now that the surface σ in our calculation is divided into several parts by a number of internal lines which are transversal to each other and to $\partial\sigma$. For each part the variation of the action yields the equations of motion on its boundary. Now we want to glue these parts together. How can the result for initial σ be restored after that? Clearly we have to add up the integrals over all the parts and demand that the variations of different parts should be equal to each other on those boundaries which are going to be identified. Then integrals over all internal lines in δS come up twice with opposite signs and cancel each other. (We know that it really *had* to be so due to Stokes' theorem.) Again we get the Hamiltonian equations of motion for the boundary trajectories of σ. It allows us to prove the theorem in its full generality. Indeed, even if we cannot embed a surface $\tilde{\sigma}$ into a contractible domain in M^{2n}, we still can divide
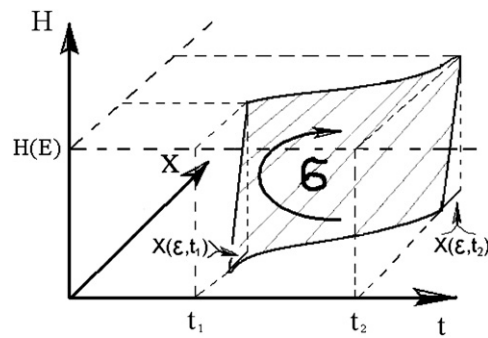
Figure 2. An energy axis is added.

it into small parts such that every part together with its nearest neighbors belongs to some contractible domain. The action functional is invariant under coordinate transformations. Each part of the surface can be varied in any coordinate system and contributions of division lines cancel each other. The final result contains only equations of motion on the boundary trajectories of $\tilde{\sigma}$. The theorem is proven.

After this analytic derivation we would like to add a nice geometric picture of the theorem. As it was already mentioned, those variations which do not change boundary trajectories are irrelevant due to Stokes' theorem. Let us consider a variation of the surface σ which is not trivial only in a small vicinity of a part of one boundary trajectory. Then the variation is just the difference between two integrals over small pieces of σ and $\tilde{\sigma} = \sigma + \delta\sigma$. And due to Stokes' theorem again this difference is equal to the integral over some surface which connects the initial trajectory and its image after the variation. So, it means that the integral of $\omega - \mathrm{d}H \wedge \mathrm{d}t$ over arbitrary thin stripe along the physical boundary trajectory should be of order $\mathcal{O}((\delta x)^2)$ with no regard to the orientation of the stripe. (Note also that we can take a not too long part of the trajectory and use the Darboux coordinates on the stripe if we like.) It means that $(\omega - \mathrm{d}H \wedge \mathrm{d}t)(l, a) = 0$ for any vector a and a vector l tangent to the trajectory. It is precisely the equations of motion.

Let us also briefly mention that we could define a 'superextended' phase space with *two* new coordinates, H and t. In this case we should perform variations only on a hypersurface $H = H(x)$ and may consider the parameter ϵ as a coordinate along the H-axis (figure 2).

5. Some additional remarks

Curiously enough, the idea of some 2-form integration over two-dimensional surface between two trajectories appeared recently in [21] which goes in a direction somewhat opposite to the lines of our work. Systems with velocity dependent forces, generally admitting neither Hamiltonian nor Lagrangian formulation (see, however, [22] and [23]), are considered in this reference in the language of positions and velocities (we would like to remind here that, in contrast, our main goal is the treatment of Hamiltonian systems for which it is generally impossible to separate coordinates and momenta). For these (generally dissipative) systems a variational principle is obtained which yields the equations of motion and some more equation on the bulk of the integration surface with unclear dynamical meaning. So that a difficult problem of joint solvability arises, see [21] for details. In spite of certain similarity, the intersection of this principle with ours is rather trivial. In the case of conservative systems it

reduces to the Lagrangian version of action (5), but the relevant systems are those for which the coordinates and momenta are clearly separated, and much simpler action principle of form (2) or (4) exists.

Note also that we could formulate the variational principle invariantly but without dealing too much with the language of exterior calculus. The obvious relation $dx^\mu = \dot{x}^\mu \, dt + x'^\mu \, d\varepsilon$ leads us to the action integral:

$$S = \int_{t_1}^{t_2} \left(\int_0^E \omega_{\mu\nu} \dot{x}^\nu x'^\mu \, d\epsilon - (H(x(E, t)) - (H(x(0, t)))) \right) dt.$$

And varying it with respect to $x^\mu(\epsilon, t)$ in the same way as in section 4, we get the same result

$$\delta S = \int_{t_1}^{t_2} (\omega_{\mu\nu} \dot{x}^\nu - \partial_\mu H)\delta x^\mu \Big|_{\varepsilon=E} dt - \int_{t_1}^{t_2} (\omega_{\mu\nu} \dot{x}^\nu - \partial_\mu H)\delta x^\mu \Big|_{\varepsilon=0} dt.$$

Then we have to mention the problem of boundary conditions again. It may seem to be even more intricate in the case of non-exact forms because if the initial points in the phase space are given one has to guess properly the final points at least for *two* trajectories (at $\epsilon = 0$ and $\epsilon = E$) for the stationary surfaces to exist. But we can easily reformulate our principle: choose in F^{2n+1} only one physical trajectory and one auxiliary line with the same initial and final points such that a nonsingular surface in the phase space exists with the boundary equal to these two curves. Then we can take our action integral (5) along the surfaces of that kind with the auxiliary line fixed and the physical trajectory free to change (except the boundary points, of course). By literally the same calculations as in section 4 it can be easily verified that all surfaces for which the equations of motion are valid on the trajectory would compose the stationary manifold for the action considered.

And a final remark deals with Maupertius principle. Suppose we consider only the first term in action (5) but vary this action only in the class of surfaces with boundary trajectories laying on two different hypersurfaces of constant Hamiltonian. The result is that for every vector l^ν tangent to the $H = $ const hypersurface in M^{2n} the equation $\omega_{\mu,\nu} \dot{x}^\mu l^\nu = 0$ should be held true on the boundary trajectories. In the phase space M^{2n} this equation uniquely defines a curve which is compatible with the equations of motion $\omega(\dot{x}, .) = -dH(.)$ because $dH(l) = 0$ for the vectors considered. But the time coordinate may be chosen arbitrarily because the equation is reparametrization invariant. So we get the invariant form of the Maupertius principle.

6. Examples and discussion

The usual action principle for Hamiltonian systems is given by (4) (or by (2) for classical systems). On exact manifolds the 1-form γ is globally well defined and action (4) can be perfectly used, but one should remember that although the choice of γ is a coordinate-free procedure, it is not unique and effectively means a kind of distinction between what we would call momenta and what we would call coordinates. For non-exact manifolds γ exists only locally, and even if with a particular choice of γ we were able to pursue a variational approach for one trajectory it does not mean yet that we would be able to repeat it for some another path without changing the set-up.

Our variational principle is completely invariant and can be used for any Hamiltonian system. Let us consider a few very simple examples. First, we take a sphere with $\omega = \cos\theta \, d\theta \wedge d\varphi$ in spherical coordinates, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \varphi \in [0, 2\pi)$ and Hamiltonian $H = \sin\theta$. (One can check that ω is non-degenerate and H is smooth near the poles $\theta = \pm\frac{\pi}{2}$, for example, by going to new coordinates $x = \cos\theta \cdot \cos\varphi, y = \cos\theta \cdot \sin\varphi$ in which

$\omega = \frac{\mathrm{d}y \wedge \mathrm{d}x}{\sqrt{1-x^2-y^2}}$ and $H = \sqrt{1-x^2-y^2}$ are obviously well defined near $x = y = 0$.) The equations of motion are $\dot\theta = 0, \dot\varphi = 1$. Let $\varphi(\epsilon, t_1) = 0, \theta(\epsilon, t_1) = \epsilon, 0 \leqslant \epsilon \leqslant E < \frac{\pi}{2}$ be a family of initial data and $\varphi(\epsilon, t_2) = \alpha(t_2 - t_1), \theta(\epsilon, t_2) = \epsilon$ – a family of final points. (We use α to show what happens if final points are 'wrong'.) For this case (one degree of freedom) irrelevance of intermediate trajectories is obvious because the surface of integration $x(\epsilon, t)$ is completely defined by boundary paths $\theta(\phi(t))$ at $\epsilon = 0$ and $\epsilon = E$: $\varphi(E, t), \varphi(0, t), \theta(E, t) = f_2(\varphi(E, t)), \theta(0, t) = f_1(\varphi(0, t))$. Action (5), after one half of the integrations have been performed, turns into the following form:

$$S = \int_0^{\alpha(t_2-t_1)} \mathrm{d}\varphi \, (\sin(f_2(\varphi)) - \sin(f_1(\varphi))) - \int_{t_1}^{t_2} \mathrm{d}t \, (\sin(f_2(\varphi(E, t))) - \sin(f_1(\varphi(0, t)))).$$

The variations of $f_2(\varphi), \varphi(E, t)$ and $f_1(\varphi), \varphi(0, t)$ should be performed independently, and actually we can even fix one of the boundary trajectories and obtain the equations of motion only for the second path. If we vary $\varphi(E, t)$ as a function of time we have $\delta_\varphi S = -\int \mathrm{d}t \cos(f_2(\varphi)) f_2' \delta\varphi$. It implies $f_2 = \text{const}$, i.e. $\dot\theta = 0$. Now we vary $f_2(\varphi)$ independently:

$$\delta_{f_2} S = \int_0^{\alpha(t_2-t_1)} \mathrm{d}\varphi \cos(f_2(\varphi)) \delta f_2(\varphi) - \int_{t_1}^{t_2} \mathrm{d}t \cos(f_2(\varphi)) \delta f_2(\varphi)$$

and after changing the variable in the first integral according to $\mathrm{d}\varphi = \dot\varphi \, \mathrm{d}t$ we get $\dot\varphi = 1$. The consistency condition ($\Delta\varphi = \int \dot\varphi \, \mathrm{d}t$) leads to $\alpha = 1$. There is no stationary point for the action S otherwise.

Actually, $\mathcal{S}^2$ is a very simple manifold and one could use 1-form $\gamma = \sin\theta \, \mathrm{d}\varphi$ on it. But this γ is singular in the poles, and if we were not so clever we could take $\tilde\gamma$ with poles on the physical trajectory. One more remark is that we could make $E \to \frac{\pi}{2}$ and convert action (4) to a rather nice form

$$S = \int_{\mathcal{D}} \omega - \int_{t_1}^{t_2} H \, \mathrm{d}t \tag{6}$$

where $t_2 - t_1 = 2\pi$ and $\mathcal{D}$ is the upper part of the sphere with a periodic trajectory as a boundary. This expression is valid only for contractible closed trajectories and is being successfully used for the purposes of symplectic topology [20, 24], but it is not well known among physicists. (The authors were unaware of it before searching the literature for historical references for this paper.) We have to note that actions (5) and (6) present even more intricate problems for the direct variational methods because if one changes the homotopy class of σ or $\mathcal{D}$ he will not affect the stationarity of the action with respect to small continuous variations of the surface but will shift the value of the action by some element of the so-called period group which consists of all the real numbers obtained by integrating ω over submanifolds homeomorphic to $\mathcal{S}^2$. Sometimes this ambiguity is only of academic interest because, for example, for systems on $\mathcal{S}^2$ it means only that we could force the disk $\mathcal{D}$ to wrap several times around the whole phase space. (Note that in this case the values of action (6) would acquire an additional constant, a multiple of the total $\mathcal{S}^2$ area. It will change nothing for the variations of (6) but no 1-form γ would be correctly defined.) But unfortunately it is not always so good, and sometimes the period group may even be everywhere dense in $\mathbb{R}$, see [20], p 228.

Needless to say, our principle works equally well for both contractible and non-contractible loops. Consider now a torus $\mathcal{T}^2$ with $\omega = \mathrm{d}\theta \wedge \mathrm{d}\varphi, \theta \in [0, 2\pi), \varphi \in [0, 2\pi), H = \sin\theta$ and equations of motion $\dot\theta = 0, \dot\varphi = \cos\theta$. (This example is oversimplified not only due to the local existence of $\gamma = \theta \, \mathrm{d}\varphi$ but also because for the torus one can use a trick [6] of considering

action (3) for the periodic functions in $\mathbb{R}^2$. But still we do not want to complicate it here.) Our periodic boundary conditions $\theta_i = \theta_f = \epsilon \in [0, E], \varphi_i = 0, \varphi_f = 2\pi n, n \in \mathbb{Z}$ imply

$$S = \int_0^{2\pi n} d\varphi \, (f_2(\varphi) - f_1(\varphi)) - \int_0^T dt \, (\sin(f_2(\varphi(E, t))) - \sin(f_1(\varphi(0, t)))).$$

Suppose that we fix the $\epsilon = 0$ path and vary only the $\epsilon = E$ trajectory. Variation of $f_2(\varphi)$ yields $f_2 = \text{const}$ and after that varying $\varphi(t)$ we get $\dot{\varphi} = \cos\theta$ with the consistency condition again $\Delta\varphi = \int \dot{\varphi} \, dt$. The last equation means that trajectories satisfying the periodicity relation $\varphi(t+T) = \varphi(t) + 2\pi n$ exist at $\cos\theta = \frac{2\pi n}{T}$. There are several different solutions to the equation $\delta S = 0$ for large enough T. (Different solutions have different boundary trajectories, and each one of these solutions also has an infinite degeneracy due to possibility of wrapping σ any number of times around the torus without changing the boundaries.) In the invariant setting we do not have to worry about the location of these stripes σ, but if we decided to use $\gamma = \theta \, d\varphi$ it would be necessary to cut the torus along the line of $\theta = 0$.

These $n = 1$ examples do not allow us to illustrate the Maupertius principle. For this purpose let us consider the simplest example of $n = 2$ system: $M^{2n} = S^2 \times \mathbb{R}^2, \omega = \cos\theta \, d\theta \wedge d\varphi + dp \wedge dq, H = \sin\theta + \frac{p^2}{2}$. The equations of motion are $\dot{\theta} = 0, \dot{p} = 0, \dot{\varphi} = 1, \dot{q} = p$. To apply the Maupertius principle we first find the surface of constant Hamiltonian: $\sin\theta + \frac{p^2}{2} = \text{const}$. After that we take the boundary conditions: $\varphi_i = 0$, $\varphi_f = t_2 - t_1, \theta_i = \theta_f = \alpha\epsilon, q_i = 0, q_f = \beta\epsilon(t_2 - t_1), p_i = p_f = \beta\epsilon$ and the abbreviated action

$$S_M = \int \omega = \int_0^{t_2-t_1} d\varphi(\sin(f_2(\varphi)) - \sin(f_1(\varphi))) + \int_0^{\beta\epsilon(t_2-t_1)} dq(g_2(q) - g_1(q))$$

where $\theta(E, t) = f_2(\varphi(E, t)), p(E, t) = g_2(q(E, t))$ and similarly for f_1, g_1. Varying the second trajectory we have $\delta S_M = \int d\varphi \cos(f_2)\delta f_2 + \int dq \, \delta g_2$. We consider $q = q(\varphi)$ and use the relation $\cos(f_2)\delta f_2 + g_2\delta g_2 = 0$ on the $H = \text{const}$ surface. The result is $\delta S_M = \int d\varphi(-g_2 + q'(\varphi))\delta g_2$ and $q'(\varphi) = p$. Independent variation of $q(\varphi)$ yields $\int d\varphi \, g_2\delta q'$ and $g_2 = \text{const}$. Due to $H = \text{const}$ it implies $f_2 = \text{const}$ and finally we have $\theta = \alpha E = \text{const}$, $p = \beta E = \text{const}, \frac{q}{\varphi} = \beta E = \text{const}$.

We have presented a new completely invariant approach to the variational formulation of Hamiltonian mechanics. Our principle can be applied to any Hamiltonian system, but for the direct methods of variational calculus it inherits all the usual mathematical difficulties related to action principles in Hamiltonian form. Therefore the question of how far one could go further with it remains an open problem of mathematical nature.

Acknowledgment

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